Hitting with Restart: A Reason for Sisyphus Labour

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Konstantin Avrachenkov[*], Alexey Piunovskiy[†],
Yi Zhang[‡]

Project-Teams Maestro

Abstract: Motivated by applications in telecommunications, computer science and physics, we consider a discrete-time Markov process with restart in the Borel state space. At each step the process either with a positive probability restarts from a given distribution, or with the complementary probability continues according to a Markov transition kernel. The main focus of the present work is the expectation of the hitting time (to a given target set) of the process with restart, for which we obtain the explicit formula. Observing that the process with restart is positive Harris recurrent, we obtain the expression of its unique invariant probability. Then we show the equivalence of the following statements: (a) the boundedness (with respect to the initial state) of the expectation of the hitting time; (b) the finiteness of the expectation of the hitting time for almost all the initial states with respect to the unique invariant probability; and (c) the target set is of positive measure with respect to the invariant probability. We illustrate our results with two examples in uncountable and countable state spaces.

Key-words: Discrete-time Markov Process with Restart, Expected Hitting Time

[*] Inria Sophia Antipolis, France, `k.avrachenkov@sophia.inria.fr`
[†] University of Liverpool, United Kingdom, `piunov@liv.ac.uk`
[‡] University of Liverpool, United Kingdom, `yi.zhang@liv.ac.uk`

Le Temps de Premier Passage dans les Processus de Markov avec Redémarrage:
Raison du Travail de Sisyphe

Résumé : Motivé par diverses applications provenant de télécommunications, informatique et de la physique, nous considerons un processus generale de Markov dans l'espace de Borel avec une possibilité de redémarrage. À chaque étape, avec une probabilité le processus redémarre a partir d'une distribution donnée et avec la probabilité complémentaire le processus continue l'évolution selon une noyau de Markov. Nous étudions l'espérance du temps de premier passage à l'ensemble donné. Nous obtenons une formule explicite pour l'espérance du temps de premier passage et démontrons que le processus avec redémarrage est Harris positif récurrent. Ensuite, nous établissons que les assertions suivantes sont équivalentes : (a) le fait d'être limité (par rapport à l'état initiale) de l'espérance du temps de premier passage; (b) la finitude de l'espérance du temps de premier passage pour presque tous les états initiaux par rapport à la probabilité invariante unique; et, (c) l'ensemble cible est de mesure positive par rapport à la probabilité invariante. Enfin, nous illustrons nos résultats théoriques avec deux exemples dans les espaces dénombrables et non dénombrables.

Mots-clés : Processus de Markov en Temps Discret, L'Espérance du Temps de Premier Passage

1 Introduction

We study a discrete-time Markov process in the Borel state space with restart. At each step the process either with the positive probability p restarts from a given distribution, or with the complementary probability $1 - p$ continues according to a Markov transition kernel. Such processes have many applications in telecommunications, computer science and physics. Let us cite just a few. Both TCP (Transmission Control Protocol) and HTTP (Hypertext Transfer Protocol) can be viewed as protocols restarting from time to time, Krishnamurthy and Rexford [13], Maurer and Huberman [16]. The PageRank algorithm, Brin and Page [8], in information retrieval models the behaviour of an Internet user surfing the web and restarting from a new topic from time to time. Markov processes with restart are useful for the analysis of replace and restart types protocols in computer reliability, Asmussen *et al.* (2008) [2], Asmussen *et al.* (2014) [3], Kulkarni *et al.* [14]. The restart policy is also used to speedup the Las Vegas type randomized algorithms Alt *et al.* [1], Luby *et al.* [15]. Finally, human and animal mobility patterns can be modeled by Markov processes that restart from some locations González *et al.* [10], Walsh *et al.* [18].

We observe that the process with restart exhibits several desirable self-stabilizing properties. For example, it is always positive Harris recurrent with the unique invariant probability (see Corollary 2.1). Moreover, for a fixed target set, when the (expected) hitting time of the original process without restart is infinite (recalling the Pólya theorem for random walks), the one of the process with restart could be still finite. The present work elaborates on this point. As a main result we show (see Theorem 2.3) the equivalence between the following statements, some of which are otherwise superficially stronger than the others.

(a) The expected hitting time of the process with restart is finite for almost all the initial states with respect to the invariant probability.

(b) The expected hitting time of the process with restart is *(uniformly) bounded* with respect to all the initial states.

(c) The target set is of positive measure with respect to the invariant probability.

This characterization result is based on the obtained explicit and simple formula of the expected hitting time of the process with restart in terms of the expected discounted hitting time of the process without restart; see Theorem 2.2, whereas the formula itself can be useful in the optimization of the expected hitting time of the process with restart with respect to the restart probability; see Section 3 for examples.

Let us mention some related work to the present one in the current literature. The continuous-time Markov process with restart was considered in Avrachenkov *et al.* [5]. According to Theorem 2.2 in [5], the continuous-time Markov process with restart is positive Harris recurrent in case the original process is honest. At the same time, the process with restart is not positive Harris recurrent if the original process is not honest (i.e., the transition kernel is substochastic; in case the state space is countable, that means the accumulation of jumps). The objective of [5] does not lie in the expected hitting time, but in the representation of the transition probability function of the (continuous-time) process with restart in terms of the one of the original (continuous-time) process without restart. This is trivial in the present discrete-time setup. Here our focus is on the characterization of the expected hitting time. We also would like to mention the two works Dumitriu *et al.* [9], Janson and Peres [12], dealing with the control theoretic formulation, where the controller decides (dynamically) whether it is beneficial or not to perform a restart at the current state. That line of research can be considered complementary to ours.

The rest of this paper is organized as follows. The description of the process with restart and the main statements are presented in Section 2, which are illustrated by two examples in Section 3. The paper is finished with a conclusion in Section 4.

2   Main statements

Let us introduce the model formally. Let E be a nonempty Borel space endowed with its Borel σ-algebra $\mathcal{B}(E)$. Consider a discrete-time Markov chain $\tilde{X} = \{\tilde{X}_t, \ t = 0, 1, \dots\}$ in the Borel state space E with the transition probability function $\tilde{P}(x, dy)$ being defined by

$$\tilde{P}(x, \Gamma) := p\nu(\Gamma) + (1-p)P(x, \Gamma),$$

for each $\Gamma \in \mathcal{B}(E)$, where $p \in (0, 1)$ and $P(x, dy)$ is a transition probability function, and ν is a probability measure on $\mathcal{B}(E)$. Let $X := \{X_t, t = 0, 1, \dots\}$ denote the Markov chain corresponding to the transition probability $P(x, dy)$. We assume the two processes X and $\tilde{X}$ are defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$; if we emphasize that the initial state is $x \in E$, then E_x denotes the corresponding expectation operator, and the notation P_x is similarly understood.

The process $\tilde{X}$ is understood as the modified version of the process X, and is obtained by restarting (independently of anything else) the process X after each transition with probability $p \in (0, 1)$ and the distribution of the state after each restart being given by ν; whereas if there is no restart after the transition (with probability $1 - p$), the distribution of the post-transition state is $P(x, dy)$ (given that the current state is x).

The following notation is used throughout this paper. Let $P^t(x, dy), \ t = 0, 1, \dots,$ be defined iteratively as follows; for each $\Gamma \in \mathcal{B}(E)$,

$$
\begin{aligned}
P^0(x, \Gamma) &:= I\{x \in \Gamma\}, \\
P^{t+1}(x, \Gamma) &:= \int_E P^t(x, dy)P(y, \Gamma).
\end{aligned}
$$

Theorem 2.1 *The process $\tilde{X}$ has an invariant probability measure $q(dy)$ given by*

$$q(\Gamma) = \int_E \sum_{t=0}^\infty p(1-p)^t P^t(y, \Gamma)\nu(dy) \tag{1}$$

for each $\Gamma \in \mathcal{B}(E)$.

Proof. Clearly, $q(dy)$ is a probability measure. We now only need to show that $q(dy)$ is invariant by verifying that

$$q(\Gamma) = \int_E q(dx)\tilde{P}(x, \Gamma)$$

for each $\Gamma \in \mathcal{B}(E)$ as follows. Indeed,

$$\int_E q(dx)\tilde{P}(x,\Gamma) = \int_E q(dx)(p\nu(\Gamma) + (1-p)P(x,\Gamma))$$

$$= p\nu(\Gamma) + (1-p)\int_E P(x,\Gamma)\sum_{t=0}^{\infty} p(1-p)^t \int_E P^t(y,dx)\nu(dy)$$

$$= p\nu(\Gamma) + \sum_{t=0}^{\infty} p(1-p)^{t+1} \int_E P^{t+1}(y,\Gamma)\nu(dy)$$

$$= p\nu(\Gamma) + \sum_{\tau=0}^{\infty} p(1-p)^{\tau} \int_E P^{\tau}(y,\Gamma)\nu(dy) - p\nu(\Gamma)$$

$$= q(\Gamma).$$

The proof is completed. $\qquad\square$

As stated in Corollary 2.1 below, $q(dy)$ is actually the unique invariant probability measure of the process $\tilde{X}$.

The process $\tilde{X}$ is ν-irreducible, i.e., for each set $\Gamma \in \mathcal{B}(E)$ satisfying $\nu(\Gamma) > 0$, it holds that $P_x(\tau_\Gamma < \infty) > 0$ for each $x \in E$; see p.87 of Meyn and Tweedie [17]. Here and below

$$\tau_\Gamma := \inf\{t = 1, 2, \cdots : \tilde{X}_t \in \Gamma\}. \tag{2}$$

As usual, $\inf \emptyset := \infty$. Such a measure ν is called an irreducibility measure. It follows from the proof of Proposition 4.2.2 of Meyn and Tweedie [17] that the measure $q(dy)$ given in Theorem 2.1 is actually the maximal irreducibility measure; that means, any irreducibility measure is absolutely continuous with respect to $q(dy)$. Throughout this work, we adopt the definitions related to a discrete-time Markov process such as the Harris recurrence and so on all from the standard reference Meyn and Tweedie [17].

In this paper we are primarily interested in the expected hitting time of the process $\tilde{X}$ to a given set $H \in \mathcal{B}(E)$, defined by

$$\eta_H := \inf\{t = 0, 1, \cdots : \tilde{X}_t \in H\}.$$

For the future reference, we put

$${}_H P^0(x,E) := I\{x \in E \setminus H\}.$$

Denote

$$V(x) := E_x[\eta_H] \tag{3}$$

and let

$${}_H P(x,\Gamma) = \begin{cases} P(x,\Gamma), & \Gamma \not\subset H, \\ 0, & \Gamma \subset H \end{cases}$$

be the taboo transition kernel with respect to the set H. Then, one can write

$$V(x) = 1 + p \int_E V(y)\nu(dy) + (1-p)\int_E V(y)_H P(x,dy), \quad \forall\, x \in E \setminus H,$$
$$V(x) = 0, \quad \forall\, x \in H. \tag{4}$$

Furthermore, it is well known that the function $V(x)$ defined by (3) is the minimal nonnegative (measurable) solution to equation (4), and can be obtained by iterations

$$V^{(n+1)}(x) = 1 + p \int_E V^{(n)}(y)\nu(dy) + (1-p) \int_E V^{(n)}(y)_H P(x, dy), \quad n = 0, 1, \dots$$

with $V^{(n)}(x) = 0$ if $x \in H$, and $V^{(0)}(x) \equiv 0$; c.f. e.g., Proposition 9.10 of Bertsekas and Shreve [7].

One can actually obtain the minimal nonnegative solution to (4) in the explicit form.

Theorem 2.2 *(a) The minimal nonnegative solution to (4) is given by the following explicit form*

$$V(x) = V_1(x) \sum_{t=0}^{\infty} \left(p \int_E V_1(y)\nu(dy) \right)^t, \quad \forall x \in E \setminus H,$$
$$V(x) = 0, \quad \forall x \in H, \tag{5}$$

where the function V_1 is given by

$$V_1(x) := \sum_{t=0}^{\infty} (1-p)^t {}_H P^t(x, E), \quad \forall x \in E \setminus H;$$
$$V_1(x) := 0, \quad \forall x \in H. \tag{6}$$

It coincides with the unique bounded solution to the equation

$$V_1(x) = 1 + (1-p) \int_E V_1(y)_H P(x, dy), \quad \forall x \in E \setminus H;$$
$$V_1(x) = 0, \quad \forall x \in H. \tag{7}$$

(b) If $q(H) > 0$, then

$$V(x) = \frac{V_1(x)}{1 - p \int_E V_1(y)\nu(dy)} < \infty, \quad \forall x \in E \setminus H,$$
$$V(x) = 0, \quad \forall x \in H. \tag{8}$$

Proof. (a) Observe that the function V_1 given by (6) represents the expected total discounted time up to the first hitting of the process X at the set H given the initial state x and the discount factor $1 - p$. It thus follows from the standard result about the discounted dynamic programming with a bounded reward that the function V_1 is the unique bounded solution to equation (7); see e.g., Theorem 8.3.6 of Hernández-Lerma and Lasserre [11].

Now by multiplying both sides of the equation (7) by the expression

$$\sum_{t=0}^{\infty} \left(p \int_E V_1(y)\nu(dy) \right)^t$$

for all $x \in E \setminus H$, it can be directly verified that the function V defined in terms of V_1 by (5) is a nonnegative solution to (4). We show that it is indeed the minimal nonnegative solution to (4) as follows.

Let $U(x) \geq 0$ be an arbitrarily fixed nonnegative solution to (4). It will be shown by induction that

$$U(x) \geq V_1(x) \sum_{t=0}^{n} \left(p \int_E V_1(y)\nu(dy) \right)^t \quad \forall n = 0, 1, \dots. \tag{9}$$

The case when $x \in H$ is trivial.

Let $x \in E \setminus H$ be arbitrarily fixed. It follows from (4) that

$$U(x) \geq 1 + (1-p) \int_E {}_H P(x, dy) = \sum_{t=0}^{1} (1-p)^t \left(\int_E {}_H P(x, dy) \right)^t \geq 1. \qquad (10)$$

If for some $n \geq 1$

$$U(x) \geq \sum_{t=0}^{n} (1-p)^t \left(\int_E {}_H P(x, dy) \right)^t, \qquad (11)$$

then by (4),

$$\begin{aligned}
U(x) \;&\geq\; 1 + (1-p) \int_E U(y) {}_H P(x, dy) \\
&\geq\; 1 + (1-p) \int_E \left(\sum_{t=0}^{n} (1-p)^t \left(\int_E {}_H P(x, dy) \right)^t \right) {}_H P(x, dy) \\
&=\; \sum_{t=0}^{n+1} (1-p)^t \left(\int_E {}_H P(x, dy) \right)^t,
\end{aligned}$$

and so (11) holds for all $n \geq 0$ and thus by (6)

$$U(x) \geq V_1(x).$$

Consequently, (9) holds when $n = 0$.

Suppose (9) holds for n, and consider the case of $n + 1$. Then from (4),

$$\begin{aligned}
U(x) \;&=\; 1 + p \int_E U(y) \nu(dy) + (1-p) \int_E U(y) {}_H P(x, dy) \\
&\geq\; \sum_{t=0}^{n+1} \left(p \int_E V_1(y) \nu(dy) \right)^t + (1-p) \int_E U(y) {}_H P(x, dy), \qquad (12)
\end{aligned}$$

where the inequality follows from the inductive supposition. Define the function W on E by

$$W(z) = \frac{U(z)}{\sum_{t=0}^{n+1} \left(p \int_E V_1(y) \nu(dy) \right)^t}, \qquad \forall \, z \in E.$$

Then by (12),

$$W(x) \geq 1 + (1-p) \int_E {}_H P(x, dy) = \sum_{t=0}^{1} (1-p)^t \left(\int_E {}_H P(x, dy) \right)^t;$$

c.f. (10). Now, based on (12), a similar reasoning by induction as to the verification of (11) for all $n \geq 0$ shows that

$$W(x) \geq \sum_{t=0}^{k} (1-p)^t \left(\int_E {}_H P(x, dy) \right)^t, \qquad \forall \, k = 0, 1, \ldots,$$

and thus $W(x) \geq V_1(x)$. This means

$$U(x) \geq V_1(x) \sum_{t=0}^{n+1} \left(p \int_E V_1(y) \nu(dy) \right)^t.$$

Thus by induction, (9) holds for all $n \geq 0$, and thus

$$U(x) \geq V(x)$$

by (5), as desired.

(b) If $q(H) > 0$, then there exists some $T > 0$ such that $\int_E P^T(y, H) \nu(dy) > 0$, meaning that there exists some $T' \leq T$ such that $\int_{E} {}_H P^{T'}(x, E) \nu(dx) < 1$. Thus,

$$0 \leq p \int_E \sum_{t=0}^{\infty} (1-p)^t {}_H P^t(y, E) \nu(dy) < 1,$$

and so the geometric series $\sum_{t=0}^{\infty} \left(p \int_E V_1(y) \nu(dy) \right)^t$ converges. The statement follows.   □

Corollary 2.1 *The process $\tilde{X}$ is positive Harris recurrent with the unique invariant probability measure $q(dy)$ given in Theorem 2.1.*

Proof. Recall the discussion below Theorem 2.1 that $q(dy)$ is the maximal irreducibility measure. For each measurable subset Γ of E satisfying $q(\Gamma) > 0$,

$$\begin{aligned} E_x[\tau_\Gamma] &= E_x[\tau_\Gamma | \tilde{X}_1 \in \Gamma] P_x(\tilde{X}_1 \in \Gamma) + E_x[\tau_\Gamma | \tilde{X}_1 \notin \Gamma] P_x(\tilde{X}_1 \notin \Gamma) \\ &\leq 1 + \int_{E \setminus \Gamma} E_y[\eta_\Gamma] \tilde{P}(x, dy) < \infty \end{aligned}$$

for each $x \in \Gamma$, where τ_Γ is defined by (2), and the last inequality follows from the boundedness of V; see Theorem 2.2(b). Thus, $P_x(\tau_\Gamma < \infty) = 1$ for each $x \in \Gamma$. By Proposition 9.1.1 of Meyn and Tweedie [17], this means the chain $\tilde{X}$ is Harris recurrent. Now by p.231 of Meyn and Tweedie [17] and Theorem 2.1, $\tilde{X}$ is positive Harris recurrent. The fact that the invariant probability measure q is the unique (σ-finite) invariant measure follows from Theorem 10.4.4 of Meyn and Tweedie [17].   □

The next corollary is immediate.

Corollary 2.2 *If $q(H) > 0$, then both $V_1(x)$ and $V(x)$ are bounded with respect to the state $x \in E$. In particular, we have*

$$V_1(x) \leq \frac{1}{p}.$$

We also have the following chain of equivalent statements, some of which are otherwise superficially stronger than the others.

Theorem 2.3 *The following statements are equivalent.*
(a) $q(H) > 0$.
(b) $V(x) = E_x[\eta_H] < \infty$ for each $x \in E$.
(c) $V(x) = E_x[\eta_H] < \infty$ for almost all $x \in E$ with respect to $q(dy)$.
(d) $\sup_{x \in E} V(x) = \sup_{x \in E} E_x[\eta_H] < \infty$.

The proof of this theorem follows from the next lemma.

Lemma 2.1 *Statements (a) and (c) in Theorem 2.3 are equivalent.*

Proof. Statement (a) implies statement (c) by Theorem 2.2(b). It remains to show that statement (c) implies statement (a). To this end, suppose for contradiction that statement (a) does not hold, i.e., $q(H) = 0$. Then it follows from (1) that

$$\int_E \sum_{t=0}^{\infty} P^t(y, H)\nu(dy) = 0.$$

Thus, there exists a measurable subset Γ of $E \setminus H$ such that

$$\nu(\Gamma) = 1 \tag{13}$$

and

$$P_x[\eta_H^X = \infty] = 1 \ \forall \ x \in \Gamma, \tag{14}$$

where

$$\eta_H^X := \inf\{t = 0, 1, \cdots : X_t \in H\}. \tag{15}$$

Let $x \in \Gamma$ be fixed. Then by (6) and (14)

$$V_1(x) = E_x \left[\sum_{t=0}^{\eta_H^X - 1} (1 - p)^t\right] = E_x \left[\sum_{t=0}^{\infty} (1 - p)^t\right] = \frac{1}{p}.$$

Since $x \in \Gamma$ is arbitrarily fixed, and $\nu(\Gamma) = 1$ by (13), we see from (5) and the fact that $V_1(y) \geq 1$ if $y \in E \setminus H$ that $V(y) = \infty$ for each $y \in E \setminus H$. Since $q(E \setminus H) = 1$, we see that statement (c) does not hold. $\square$

Proof of Theorem 2.3. From Theorem 2.2(b), we see that (a) implies (b), which implies (c). From Lemma 2.1, (c) implies (a). Clearly, (d) implies (b). Finally, (a) implies (d) because $V_1(x)$ is bounded; see Corollary 2.2. $\square$

Next let us consider the dependance of $V(x)$ on the restart probability p for each fixed $x \in E$. When we emphasize the dependance on p, we explicitly write $V(x, p)$ and $V_1(x, p)$. In the above, $p \in (0, 1)$ was fixed. Now we formally put

$$V(x, 0) := \sum_{t=0}^{\infty} {}_H P^t(x, E) \in [1, \infty], \forall \ x \in E \setminus H;$$

$$V(x, 0) := 0, \forall \ x \in H,$$

which represents the expected hitting time of the process without restart, and

$$V(x, 1) := \frac{1}{\nu(H)} \in [1, \infty], \forall \ x \in E \setminus H;$$

$$V(x, 1) := 0, \forall \ x \in H,$$

which represents the expected hitting time of the process that restarts with full probability at each transition. Here and below, $\frac{c}{0} := \infty$ for any $c > 0$. As usual, the continuity of $V(x, p)$ at $p = a$ means $\lim_{p \to a} V(x, p) = V(x, a) \in [-\infty, \infty]$.

The next fact will be frequently referred to, and is a well known consequence of the dominated convergence theorem; we recall it here for completeness.

Lemma 2.2 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose the real valued function $f(\omega, p)$ is continuous in $p \in [a, b] \subseteq \mathbb{R}$ for each fixed $\omega \in \Omega$; is differentiable in $p \in (a, b)$ for each fixed $\omega \in \Omega$; and is integrable in ω for each fixed $p \in (a, b) \subseteq \mathbb{R}$. If there is an integrable function g on Ω such that for each $\omega \in \Omega$, $|\frac{\partial f}{\partial p}(p, \omega)| \leq g(\omega)$ for each $p \in (a, b)$, then $\int_{\Omega} f(\omega, p)\mu(d\omega)$ is differentiable in $p \in (a, b)$, and*

$$\frac{d}{dp} \int_{\Omega} f(\omega, p)\mu(d\omega) = \int_{\Omega} \frac{\partial f}{\partial p}(\omega, p)\mu(d\omega).$$

Theorem 2.4 *Suppose*

$$\int_E \sum_{t=0}^{\infty} P^t(y, H)\nu(dy) > 0. \tag{16}$$

Then the function $V(x, p)$ is infinitely many times differentiable in $p \in (0, 1)$ and is continuous in $p \in [0, 1]$. As a consequence, the problem

$$V(x, p) \to \min_{p \in [0,1]} \tag{17}$$

is solvable.

Remark 2.1 *The condition (16) in the above statement is equivalent to $q(H) > 0$ for some and then all $p \in (0, 1)$ by (1).*

Proof of Theorem 2.4. If $x \in H$, the statement holds trivially since $V(x, p) = 0$ for each $p \in [0, 1]$. Consider now $x \in E \setminus H$. Applying Lemma 2.2, one can show that $V_1(x, p)$ (as given by (6)) and $(1 - p \int_E V_1(y, p)\nu(dy))^{-1}$ are both infinitely many times differentiable in $p \in (0, 1)$. It follows from this and (8) that $V(x, p)$ is infinitely many times differentiable in $p \in (0, 1)$. For the the continuity of $V(x, p)$ at $p = 0$, it holds that

$$1 - p \int_E V_1(y, p)\nu(dy) = 1 - p \int_E E_y \left[\sum_{t=0}^{\eta_H^X - 1} (1-p)^t\right] \nu(dy) = 1 - \int_E E_y \left[1 - (1-p)^{\eta_H^X}\right] \nu(dy)$$

$$= \int_E E_y \left[(1-p)^{\eta_H^X}\right] \nu(dy) \to 1$$

as $p \to 0$ by the monotone convergence theorem; recall (15) for the definition of η_H^X. It follows from this fact, (6), (8) and the monotone convergence theorem that

$$\lim_{p \to 0} V(x, p) = \frac{\lim_{p \to 0} V_1(x, p)}{\lim_{p \to 0} \left\{1 - p \int_E V_1(y, p)\nu(dy)\right\}} = V(x, 0)$$

as desired. The continuity of the function $V(x, p)$ at $p = 1$ can be similarly established. The last assertion is a well known fact; see e.g., Bertsekas and Shreve [7]. $\square$

3   Examples

3.1   Uni-directional random walk on the line

Let $E = \mathbb{R}$, $H = [a, b]$ with $a < b$ being two real numbers. The process X only moves to the right, and the increments of each of the transitions are i.i.d. exponential random variables with

the common mean $\frac{1}{\mu} > 0$. The restart probability is denoted as $p \in (0, 1)$ as usual, and the restart distribution ν is arbitrary. Below by using Theorem 2.2 we provide the explicit formula for the expected hitting time at H of the restarted process $\tilde{X}$. (Clearly, if the initial state is outside H, then the expected hitting time of the process X at the set H is infinite.)

We can give the following informal description of this example. There is a treasure hidden in the interval $[a, b]$ and one tries to find the treasure. Once the searcher checks one point in the interval $[a, b]$, he finds the treasure. The searcher has the means only to stop and to check points between the exponentially distributed steps. This models the cost of checking frequently. It is also natural to restart the search from some base. Intuitively, by restarting too frequently, the searcher spends most of the time near the base and does not explore the area sufficiently. On the other hand, restarting too seldom leads the search to very far locations where the searcher spends most of the time for nothing. Hence, intuitively there should be an optimal value for restarting probability.

One can verify that in this example the unique bounded solution to (7) is given by $V_1(x) = 0$ for each $x \in [a, b]$,

$$V_1(x) = \frac{1}{p} \tag{18}$$

for each $x > b$,

$$V_1(x) = \frac{1}{p} - \frac{1 - p}{p} \left(1 - e^{-\mu(b-a)}\right) e^{-\mu(a-x)p} \tag{19}$$

for each $x < a$. In fact, this can be conveniently established using the following probabilistic argument. Recall that $V_1(x)$ represents the expected total discounted time up to the hitting of the set H by the process X; see (6). So for (18) one merely notes that with the initial state $x > b$, $\eta_H^X = \infty$, where η_H^X is defined by (15). For (19), one can write for each $x < a$ that

$$V_1(x) \quad = \quad E_x \left[\sum_{t=0}^{\eta_H^X - 1} (1 - p)^t \right] = E_x \left[E_x \left[\sum_{t=0}^{\eta_H^X - 1} (1 - p)^t \middle| \eta_H^X \right] \right].$$

Now the expected hitting time of the restarted process $\tilde{X}$ to the set $H = [a, b]$ is given by

$$V(x) = \left(\frac{1}{p} - \frac{1 - p}{p} \left(1 - e^{-\mu(b-a)}\right) e^{-\mu(a-x)p} \right) \sum_{t=0}^{\infty} \left(p \int_E V_1(y)\nu(dy) \right)^t,$$

with the initial state $x < a$, and by

$$V(x) = \frac{1}{p} \sum_{t=0}^{\infty} \left(p \int_E V_1(y)\nu(dy) \right)^t,$$

with the initial state $x > b$, recall (5). If the restart distribution ν is not concentrated on (b, ∞), then $q(H) > 0$, and by Theorem 2.2(b) we have

$$V(x) = \frac{\frac{1}{p} - \frac{1-p}{p} \left(1 - e^{-\mu(b-a)}\right) e^{-\mu(a-x)p}}{1 - p \int_E V_1(y)\nu(dy)},$$

with the initial state $x < a$ and by

$$V(x) = \frac{1}{p \left(1 - p \int_E V_1(y)\nu(dy)\right)},$$

for each $x > b$. In particular, if the process restarts from a single point $r < a$, the above expressions can be specified to

$$V(x) = \frac{1 - (1 - p)(1 - e^{-\mu(b-a)})e^{-\mu(a-x)p}}{p(1 - p)(1 - e^{-\mu(b-a)})e^{-\mu(a-r)p}},$$

for the initial state $x < a$ and to

$$V(x) = \left(\frac{1}{1 - e^{-\mu(b-a)}}\right)\left(\frac{1}{p(1 - p)e^{-\mu(a-r)p}}\right),$$

for each $x > b$. For the latter case $(x > b)$, by standard analysis of derivatives, one can find the optimal value of the restart probability minimizing the expected hitting time of the process with restart in a closed form, as given by

$$p_{opt} = \frac{2}{2 + \mu(a - r) + \sqrt{4 + \mu^2(a - r)^2}}.$$

Now we can make several observations: the first somewhat interesting observation is that in the case when the initial state is to the right of the interval $[a, b]$, the value of the optimal restart probability does not depend on the length of the interval but only on the average step size and on the restart position. The second observation is that when $\mu(a - r)$ is small, i.e., when either the average step size is large or the restart position is close to H, the optimal restart probability is close to $1/2$. Thirdly, when $\mu(a - r)$ is large, the optimal restart probability is small and reads

$$p_{opt} = \frac{1}{1 + \mu(a - r)} + o\left(\frac{1}{\mu(a - r)}\right).$$

3.2   Random walk on the one dimensional lattice

Let us consider a symmetric random walk on the one dimensional lattice which aims to hit $H = \{0\}$ with restart at some node r. Assume without loss of generality that the restart state r is on the positive half-line, i.e., $r > 0$.

From Theorem 2.2 we conclude that it is sufficient to solve the following equations

$$V_1(k) = 1 + \frac{1 - p}{2}[V_1(k - 1) + V_1(k + 1)], \quad k \neq 0,$$
$$V_1(0) = 0.$$

Following the standard approach for solution of difference equations, we obtain

$$V_1(k) = c\alpha_1^k + \frac{1}{p},$$

where $\alpha_1 < 1$ is the minimal solution to the characteristic equation

$$\alpha = \frac{1 - p}{2}[1 + \alpha^2],$$

and the constant $c = -\frac{1}{p}$ comes from the condition $V_1(0) = 0$. Consequently,

$$V(k) = \frac{V_1(k)}{1 - pV_1(r)} = \frac{1 - \alpha_1^k}{p\alpha_1^r}.$$

An elegant analysis can be done for the limiting case when the initial position k goes large, and hence we now minimize $\lim_{k\to\infty} V(k) = 1/(p\alpha_1^r)$, or equivalently, maximize $p\alpha_1^r$ with respect to $p \in (0,1)$. This leads to the following equation for the optimal restart probability

$$\frac{p}{\alpha_1} \frac{d\alpha_1}{dp} = -\frac{1}{r}. \tag{20}$$

Indeed, we note that

$$\lim_{p\to 0} \frac{p}{\alpha_1} \frac{d\alpha_1}{dp} = 0, \quad \lim_{p\to 1} \frac{p}{\alpha_1} \frac{d\alpha_1}{dp} = -\infty,$$

and

$$\frac{d}{dp}\left(\frac{p}{\alpha_1}\frac{d\alpha_1}{dp}\right) = -\frac{1}{\sqrt{1-(1-p)^2}}\frac{1-(1-p^2)(1-p)}{(1-p)^2(1-(1-p)^2)} < 0.$$

Thus, the left hand side of (20) is a monotone function decreasing from zero to minus infinity. Consequently, the unique solution of equation (20) is the global minimizer of $1/(p\alpha_1^r)$. The equation (20) can be transformed to the polynomial equation

$$\frac{1}{r^2}(1-p)^2(2-p) = p$$

Consider the case of large r. This is a so-called case of singular perturbation, as the small parameter $1/r^2$ is in front of the largest degree term, Avrachenkov *et al.* [4], Baumgärtel [6]. It is not difficult to see that for large values of r, the equation has one real root that can be expanded as

$$p_{opt} = \frac{c_1}{r^2} + \frac{c_2}{r^4} + ... \tag{21}$$

and two complex roots that move to infinity as $r \to \infty$. By substituting the series (21) into the polynomial equation, we can identify the terms $c_i, i = 1, 2, ...$ Thus, we obtain

$$p_{opt} = \frac{2}{r^2} - \frac{10}{r^4} + \mathrm{o}(r^{-4}).$$

4 Conclusion

In conclusion, for a discrete-time Markov process in the Borel state space with restart, we showed that the following statements are equivalent: (a) its expected hitting time to the target set is uniformly bounded with respect to the initial states; (b) its expected hitting time to the target set is finite for almost all the initial states with respect to the invariant probability; and (c) the target set is of positive measure with respect to the invariant probability. In case these statements hold, we also established the existence of an optimal restart probability in minimizing the expected hitting time to the target set. Finally, two examples were provided to illustrate our results.

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